

Quarks, mesons and (exact) flow equations*

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Abstract

Dynamical chiral symmetry breaking is described within the linear sigma model of QCD coupled to quarks. The main technical tool used for this intrinsically non-perturbative problem is an exact renormalization group equation for the quantum effective action. It is demonstrated that realistic values for phenomenological quantities like the pion decay constant, constituent quark masses or the chiral condensate are obtainable.

1 Chiral symmetry breaking in QCD

The strong interaction dynamics of quarks and gluons is widely believed to be described by quantum chromodynamics (QCD). One of its most striking features is asymptotic freedom [1] which makes perturbative calculations reliable in the high energy regime. On the other hand, the increase in strength of the gauge coupling as one lowers the relevant momentum scale is assumed to be the cause of confinement. As a consequence, the low-energy degrees of freedom in strong interaction physics are mesons, baryons and glueballs rather than quarks and gluons.

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Chiral symmetry breaking (χ SB) as one of the most prominent features of strong interaction dynamics is a phenomenologically well established fact (see, e.g., [2]). Yet, a rigorous field theoretic description of this phenomenon in four dimensional space–time starting from first principles is still missing. The classical QCD Lagrangian does not couple left– and right–handed quarks in the chiral limit (vanishing current quark masses). It therefore exhibits in addition to the local $SU(3)$ color symmetry a global chiral invariance under $U_L(N) \times U_R(N) = SU_L(N) \times SU_R(N) \times U_V(1) \times U_A(1)$ where N denotes the number of massless quark flavors q :

$$\begin{aligned} q_R &\equiv \frac{1 - \gamma_5}{2} q \longrightarrow \mathcal{U}_R q_R ; \quad \mathcal{U}_R \in U_R(N) \\ q_L &\equiv \frac{1 + \gamma_5}{2} q \longrightarrow \mathcal{U}_L q_L ; \quad \mathcal{U}_L \in U_L(N) . \end{aligned} \tag{1}$$

However, the axial Abelian subgroup $U_A(1) = U_{L-R}(1)$ is spontaneously broken in the quantum theory by an anomaly of the axial–vector current. This breaking proceeds without the occurrence of a Goldstone boson coupling to the gauge invariant $U_A(1)$ current [3]. The $U_V(1) = U_{L+R}(1)$ subgroup corresponds to baryon number conservation and remains unaffected. The remaining chiral $SU_L(N) \times SU_R(N)$ group appears to be spontaneously broken to the diagonal (generalized) isospin subgroup $SU_{L+R}(N)$ by the QCD dynamics

$$SU_L(N) \times SU_R(N) \longrightarrow SU_{L+R}(N) = SU_V(N) . \tag{2}$$

This is reflected in the light meson spectrum by the existence of eight relatively light parity–odd (pseudo–)Goldstone bosons: π^0 , π^\pm , K^0 , \bar{K}^0 , K^\pm and η . Their comparably small masses are a consequence of the explicit χ SB due to small but non–vanishing current quark masses.

Mesons are thought of as (color neutral) quark–antiquark bound states $\varphi^{ab} \sim \bar{q}_L^a q_R^b$, $a, b = 1, \dots, N$, which therefore transform under chiral rotations (1) as

$$\varphi \longrightarrow \mathcal{U}_L^\dagger \varphi \mathcal{U}_R . \tag{3}$$

Hence, the χ SB pattern (2) is realized if the meson potential develops a VEV

$$\langle \varphi^{ab} \rangle = \sigma_0 \delta^{ab} ; \quad \sigma_0 \neq 0 . \tag{4}$$

One of the most crucial and yet unsolved problems of strong interaction dynamics is to derive an effective field theory for the mesonic degrees of freedom directly from QCD which exhibits this behavior.

2 A qualitative picture

We do not aim here at really bridging this gap between QCD and effective meson theories describing the infrared (IR) behavior of strong interactions. We will rather focus on a simple

model which describes many features of χ SB in QCD: the Nambu–Jona-Lasinio model [4] and extensions of it (see, e.g., [5] and references therein). The basic idea is that gluonic interactions induce effective (non-local) four-fermion interactions of the form $G(\bar{q}q)^2$. One might imagine to completely integrate out the gluons in the QCD path integral. The result would be a highly non-trivial effective action for the quarks containing an infinite set of non-local multi-quark operators. Expanding the corresponding effective Lagrangian in powers of derivatives and the quark fields the first terms would contain the standard quark kinetic term plus all possible four-fermi couplings compatible with Lorentz invariance and chiral symmetry. In particular, after appropriately Fierz rearranging the occurring spin-flavor structures one would find

$$\Gamma_{\text{eff}} = \int d^4x \left\{ Z_q \bar{q} i \not{\partial} q + \frac{G}{2} \left[(\bar{q}_a q^b) (\bar{q}_b q^a) - (\bar{q}_a \gamma_5 q^b) (\bar{q}_b \gamma_5 q^a) \right] + \dots \right\} . \quad (5)$$

Here summation over N_c quark colors is implicit. The indices a, b denote different quark flavors and run from 1 to N . The coupling constant G will be a function of the strong gauge coupling α_s and is assumed to grow as the momentum scale is lowered. Once it becomes strong enough to form $\bar{q}q$ bound states, say at some *compositeness scale* k_φ , it appears to be preferable to describe the dynamics in terms of mesons and quarks instead of quarks alone where we assume that the scale k_φ is well above the confinement scale. This can be achieved by inserting the identities

$$\begin{aligned} 1 &\sim \int \mathcal{D}\sigma_1 \exp \left\{ -\frac{1}{2} \int d^4x [\sigma_{1ab}^* + G \bar{q}_b \gamma_5 q^a] \frac{1}{G} [\sigma_1^{ab} + G \bar{q}^a \gamma_5 q^b] \right\} \\ 1 &\sim \int \mathcal{D}\sigma_2 \exp \left\{ -\frac{1}{2} \int d^4x [\sigma_{2ab}^* + iG \bar{q}_b q^a] \frac{1}{G} [\sigma_2^{ab} + iG \bar{q}^a q^b] \right\} \end{aligned} \quad (6)$$

into the QCD path integral and defining φ by

$$\sigma_1^T =: \frac{1}{2} (\varphi + \varphi^\dagger) , \quad \sigma_2^T =: -\frac{i}{2} (\varphi - \varphi^\dagger) . \quad (7)$$

This “trick” removes the four-fermi interactions for the price of introducing collective degrees of freedom φ, φ^\dagger with mass term and Yukawa interaction to the quarks but no kinetic term or self-interactions. Defining the $SU_L(N) \times SU_R(N)$ invariants

$$\begin{aligned} \rho &= \text{tr } \varphi^\dagger \varphi \\ \tau_2 &= \frac{N}{N-1} \left[\text{tr } (\varphi^\dagger \varphi)^2 - \frac{1}{N} \rho^2 \right] \\ \xi &= \det \varphi + \det \varphi^\dagger \end{aligned} \quad (8)$$

this leads to an effective action for quarks and mesons of the form

$$\begin{aligned} \Gamma_{\text{eff}} &= \int d^4x \left\{ Z_q \bar{q}_a i \not{\partial} q^a + Z_\varphi \text{tr} [\partial_\mu \varphi^\dagger \partial^\mu \varphi] + \bar{m}^2 \rho + \frac{1}{2} \bar{\lambda}_1 \rho^2 + \frac{N-1}{4} \bar{\lambda}_2 \tau_2 \right. \\ &\quad \left. - \frac{1}{2} \bar{\nu} \xi + \bar{h} \bar{q}^a \left(\frac{1+\gamma_5}{2} \varphi_a^b - \frac{1-\gamma_5}{2} (\varphi^\dagger)_a^b \right) q_b + \dots \right\} \end{aligned} \quad (9)$$

with *compositeness conditions*

$$\begin{aligned}
\overline{m}^2(k_\varphi) &= \frac{1}{2G(k_\varphi)} \\
\overline{h}(k_\varphi) &= 1 \\
Z_\varphi(k_\varphi) &= \overline{\lambda}_i(k_\varphi) = 0 ; \quad i = 1, 2 .
\end{aligned} \tag{10}$$

Moreover, the quark wave function renormalization Z_q is set to one at the scale k_φ for convenience. Note that we have included an explicit $U_A(1)$ breaking term $\overline{\nu}\xi$ by hand which mimics the effect of the chiral anomaly of QCD to leading order in an expansion of the effective potential in powers of φ . The additional $U_A(1)$ breaking invariant $\omega = i(\det \varphi - \det \varphi^\dagger)$ is CP violating and will therefore be omitted. At this point three remarks are in order:

- At the compositeness scale k_φ the meson field φ is merely an auxiliary field without kinetic term ($Z_\varphi(k_\varphi) = 0$). As the Yukawa coupling modifies the meson propagator through quark loops a scalar kinetic term is generated at scales lower than k_φ . One should, however, notice that the compositeness condition (10) for Z_φ only holds to leading order in the derivative expansion of the four-fermi interaction. In reality there will be corrections of order ∂^2 corresponding to a presumably small kinetic term for φ . Similarly, there will be corrections to the condition $\overline{\lambda}_i(k_\varphi) = 0$ due to higher dimensional quark-antiquark operators.
- The effective potential U_k of Γ_{eff} is purely quadratic in φ at the scale k_φ . Therefore $\langle \varphi \rangle = 0$ and there is no χ SB. This means that there are mesonic bound states at the compositeness scale even without χ SB!
- We have refrained here for simplicity from considering four-quark operators with vector and pseudo-vector spin structure. Their inclusion is straightforward and would lead to vector mesons in the effective action (9).

The question remains how chiral symmetry could possibly be broken within this model. It is suggestive to try to answer this question by following the evolution of the effective potential U_k from k_φ to lower scales using renormalization group (RG) methods. The hope would be that U_k develops a minimum away from the origin at some scale $k < k_\varphi$ such that $\langle \varphi \rangle = \sigma_0 \mathbf{1} \neq 0$. In the far IR ($k \rightarrow 0$) one could then extract the (renormalized) VEV and relate it to phenomenological quantities like the pion decay constant f_π , the chiral condensate, the constituent quark mass or the various meson masses. However, there are also several “input parameters” at the scale k_φ which are necessary to fix the RG boundary conditions for the (dimensionless) renormalized couplings

$$\begin{aligned}
\epsilon(k) &= k^{-2} m^2(k) = \overline{m}^2(k) Z_\varphi^{-1}(k) k^{-2} \\
h^2(k) &= \overline{h}^2(k) Z_\varphi^{-1}(k) Z_q^{-2}(k) \\
\lambda_i(k) &= \overline{\lambda}_i Z_\varphi^{-2}(k) ; \quad i = 1, 2 \\
\nu(k) &= \overline{\nu}(k) Z_\varphi^{-\frac{N}{2}}(k) k^{N-4} .
\end{aligned} \tag{11}$$

Hence, the question arises how much predictive power there is in this model. Before trying to give an answer we note that due to the small scalar wave function renormalization Z_φ at the scale k_φ at least the Yukawa coupling is large. This implies that a perturbative RG analysis of the effective potential would not be reliable and we have to resort to non-perturbative methods. A frequent choice for the NJL-model are large- N_c or mean field techniques which have been used, e.g., to calculate the RG-improved effective potential and study the order of the chiral phase transition within this model in [6]. However, $N_c = 3$ is not a large number and one would prefer to have a quantitatively more accurate method available.

3 An exact renormalization group equation

Exact renormalization group equations (ERGEs) have a long history [7] and have been formulated in many different but related ways. Their use in field theory has been limited due to the difficulty of solving them, even approximately, in a systematic way. We will follow here the approach of Wetterich [8] which seems particularly suited for practical calculations. A treatment of the quark meson model introduced in the last section along this line can be found in [9] and will be described in the remainder of this talk. The basic idea is an implementation of the Kadanoff–Wilson block–spin RG in the continuum. Consider, e.g., a real scalar field χ^a (a labeling internal degrees of freedom) in d (Euclidean) space–time dimensions with classical action $S_{\text{cl}}[\chi]$. We define

$$\begin{aligned} S_k[\chi, J] &= S_{\text{cl}}[\chi] + \Delta S_k[\chi] - \int d^d x J_a(x) \chi^a(x) \\ \Delta S_k[\chi] &= \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} R_k(q^2) \chi_a(-q) \chi^a(q) . \end{aligned} \tag{12}$$

Here $R_k(q^2)$ denotes an appropriately chosen (see below) IR cutoff function and J are the usual scalar sources introduced to define generating functionals. We require that $R_k(q^2)$ becomes infinitesimally small for $q^2 \gg k^2$ whereas for $q^2 \ll k^2$ it should behave as $R_k(q^2) \simeq k^2$. This means that all Fourier components of χ^a with momenta smaller than the IR cutoff k should acquire an effective mass $m_{\text{eff}} \simeq k$ and therefore decouple while the high momentum components of χ^a should not be affected by R_k . Hence, if we define the generating functional of connected Green functions

$$W_k[J] = \ln \int \mathcal{D}\chi \exp\{-S_k[\chi, J]\} \tag{13}$$

only Fourier components of χ^a with momenta $q^2 \gtrsim k^2$ will be integrated out. Defining the effective action or generating functional of 1PI Green functions

$$\Gamma_k[\varphi] = -W_k[J] + \int d^d x J_a(x) \chi^a(x) - \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} R_k(q^2) \varphi_a(-q) \varphi^a(q) \tag{14}$$

with classical fields

$$\varphi^a \equiv \langle \chi^a \rangle = \frac{\delta W_k[J]}{\delta J_a} \quad (15)$$

it is straightforward to show [8] that

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \text{Tr} \left\{ \left[\Gamma_k^{(2)}[\varphi] + R_k \right]^{-1} \partial_t R_k \right\} . \quad (16)$$

Here $t = \ln(k/\Lambda)$ with some arbitrary momentum scale Λ , and $\Gamma_k^{(2)}$ denotes the *exact* inverse propagator

$$\left[\Gamma_k^{(2)} \right]_{ab}(q, q') = \frac{\delta^2 \Gamma_k}{\delta \varphi^a(-q) \delta \varphi^b(q')} . \quad (17)$$

The trace in (16) is taken over all internal indices and also involves a momentum integration. The ERGE (16) is a functional differential equation for Γ_k which can be viewed as a partial differential equation for the infinitely many variables $\varphi^a(q)$ and t . A solution (exact or approximate) can only be obtained once we specify appropriate boundary conditions. This is most conveniently done by fixing all of Γ_k at a single scale which we choose to be Λ . Hence, Λ is nothing but a renormalization scale. For practical calculations one will often take Λ to be in the UV and use (16) to evolve Γ_k towards the IR. One can show [8] that

$$\begin{aligned} \lim_{k \rightarrow 0} R_k(q^2) = 0 & \Rightarrow \lim_{k \rightarrow 0} \Gamma_k[\varphi] = \Gamma[\varphi] \\ \lim_{k \rightarrow \infty} R_k(q^2) = \infty & \Rightarrow \lim_{k \rightarrow \infty} \Gamma_k[\varphi] = S_{\text{cl}}[\varphi] , \end{aligned} \quad (18)$$

i.e., Γ_k interpolates between the classical (bare) action and the full quantum effective action $\Gamma[\varphi]$ at $k = 0$. A convenient choice for R_k which fulfills (18) and will be used in the following is

$$R_k(q^2) = \frac{Z_\varphi q^2 e^{-q^2/k^2}}{1 - e^{-q^2/k^2}} \quad (19)$$

where Z_φ denotes the scalar wave function renormalization constant. We wish to stress that (16) is an *exact* equation for the full effective action Γ_k , i.e. the generating functional of all 1PI Green functions, where only quantum fluctuations with momenta $q^2 \gtrsim k^2$ have been integrated out. It can be generalized to contain fermions [10, 9] as well as gauge fields [11]. Another important feature of Γ_k is that it is IR and ultraviolet (UV) finite, the former being obvious due to the presence of an IR cutoff. Technically, UV finiteness is a consequence of the factor $\partial_t R_k$ in (16). For the choice (19) the arguments of all momentum integrals will be suppressed exponentially in the UV. Intuitively it may be understood by recalling that (16) is used to evolve Γ_k from a given scale k down to the IR. Hence, at k all quantum fluctuations with momenta $q^2 \gtrsim k^2$ are assumed to be integrated out already. As a consequence, all momentum integrations are limited to a small range around k and, in particular, cut off the UV momentum range.

Even though a variation of $t = \ln \frac{k}{\Lambda}$ can be viewed as a change of the UV scale Λ for fixed k , we will adopt here the technically equivalent but conceptually opposite point of

view that it rather corresponds to a change of the IR scale k for fixed Λ . Eq. (16) may then be interpreted as a “microscope” with variable “resolution” k . Starting from high (short distance) resolution $k = \Lambda$ where the dynamics of a system is known and described by Γ_Λ , eq. (16) then permits to follow the change of the action as one continuously lowers the scale k thereby embracing larger and larger structures of size $\sim k^{-1}$.

How can we apply this approach to the study of strong interaction dynamics? One would want to start from a high enough resolution $k = \Lambda$ where the microscopic dynamics is governed by the QCD Lagrangian with quarks and gluons as fundamental degrees of freedom. As the resolution k is lowered, new degrees of freedom, i.e. mesons, baryons and possibly glueballs, will appear around or below some scale $k_\varphi \simeq 700$ MeV. The corresponding transition to such collective degrees of freedom can in principle be described within the framework of the ERGE [12]. Between k_φ and the confinement scale, Γ_k will describe the dynamics of quarks, gluons and hadrons. Only for scales below the confinement scale the dynamical degrees of freedom will solely be hadrons. We will not attempt here to carry out this ambitious program completely which should ultimately lead to a determination of IR quantities like hadron masses, f_π or the chiral condensate in terms of α_s and the quark masses only. We will rather focus on the range of scales $k \lesssim k_\varphi$ and therefore take $\Lambda = k_\varphi$ and $\Gamma_\Lambda = \Gamma_{\text{eff}}$ of (9) in combination with the compositeness conditions (10). It should be noted, though, that (9) is certainly a rather crude approximation to the full QCD effective action at scales around k_φ as far as degrees of freedom are concerned. In principle, baryons, glueballs and other bound states might already exist at scales $k \simeq k_\varphi$. Yet, we are not aiming here at the full IR limit of QCD. One may hope that the inclusion of additional degrees of freedom beyond (9) is not crucial for an understanding of χ SB in the light mesonic sector of QCD. Ultimately this can, however, only be decided in view of the correctness (or incorrectness) of the results for phenomenological IR quantities obtained from (9).

Even though (16) is an exact equation we are still far from being able to *solve* it exactly for any realistic $4d$ QFT. In fact, the mere existence of an exact equation describing a given QFT is not too surprising. The difficult task is rather to find an approximation scheme which is technically manageable but also sophisticated enough to allow for the computation of some interesting IR quantities with reasonable accuracy. The main difficulty here is that even if one starts with a Γ_k containing only a finite set of operators at the scale Λ (e.g., the classical action), an infinitesimal change of scale governed by (16) will generate the full infinite set of operators which are consistent with the symmetries of the theory under consideration. The main idea here is therefore to try to identify a finite set of operators in the effective action which is *approximately closed* under a change of scale and in addition captures at least some of the physically interesting IR quantities. To be specific, we will try to attack the problem at hand by truncating Γ_k in such a way that it contains all naively relevant and marginal operators, i.e. those with canonical dimensions $d_c \leq 4$ in four space-time dimensions. This means that we will take $\Gamma_k = \Gamma_{\text{eff}}$ with Γ_{eff} defined in (9) and ignore the evolution and effects coming from operators with $d_c > 4$. More precisely, we will use (9) only in the symmetric regime, i.e. for $m^2 > 0$. Once the system crosses into the broken regime and φ develops a non-vanishing VEV σ_0 , we will instead expand

the effective potential around σ_0 also up to operators of canonical dimension four:

$$U_k = \frac{1}{2}\bar{\lambda}_1(k) \left[\rho - N\sigma_0^2(k) \right]^2 + \frac{N-1}{4}\bar{\lambda}_2(k)\tau_2 + \frac{1}{2}\bar{\nu}(k) \left[\sigma_0^{N-2}(k)\rho - \xi \right]. \quad (20)$$

We emphasize that truncating higher dimensional operators does not imply that one has to assume that the corresponding coupling constants are small. In fact, this could only be expected as long as the relevant and marginal couplings are small as well. Taking the simplest ones into account indeed shows that they can be quite large, in general. What is required, though, is that their *influence* on the evolution of those couplings kept in the truncation, for instance, the set of equations (22) below, is small.

One should add that higher dimensional operators can by no means always be neglected, the most prominent example being QCD itself. It is the very assumption of our treatment of χ SB that the momentum dependence of the coupling constants of some six-dimensional quark operators $(\bar{q}q)^2$ develop poles in the s -channel indicating the formation of mesonic bound states. One might thus ask for a guiding principle which would allow to tell a priori which operators to keep and which to throw away. Unfortunately there is no systematic criterion known at the moment. It is therefore mainly a matter of physical intuition which operators one decides to include. A good example are the above mentioned $(\bar{q}q)^2$ operators which (including the momentum dependence of their couplings) should contain a large part of the information required to describe the formation of mesonic bound states [12]. On the other hand, one might hope that, e.g., φ^6 or φ^8 operators are not really necessary to understand the properties of the potential in a small neighborhood around its minima. The ultimate check of these assumptions will, however, be the comparison of IR observables like meson masses and decay constants with their phenomenological values. In addition there are quite encouraging results for the $O(N)$ model in two, three and four dimensions [13]. It was, in particular, possible to compute critical exponents in $3d$ with a few percent precision or to describe the Kosterlitz–Thouless phase transition in $2d$ with truncations similar to the one proposed here. It is also interesting to note that the truncation (9) includes in the limit of small couplings and masses the known leading order result of the large- N_c expansion of the $U_L(N) \times U_R(N)$ model [6] for \bar{h}^2 , $\bar{\lambda}_1$ and $\bar{\lambda}_2$. This should provide at least some minimal control over this truncation, even though we hope that our results are significantly more accurate than $1/N_c$.

Inserting the truncation (9) or (20) into (16) and neglecting all operators of canonical dimension $d_c > 4$ on the right hand side reduces this partial differential equation for infinitely many variables to a finite set of ordinary differential equations. This yields, in particular, the beta functions for the couplings $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\nu}$ and \bar{m}^2 or σ_0 . Details of the calculation can be found in [9]. We will refrain here from presenting the full set of flow equations but rather illustrate the main results with a few examples. Using (11) and defining the dimensionless, renormalized VEV

$$\kappa = k^{2-d} N \sigma_R^2 = Z_\varphi k^{2-d} N \sigma_0^2 \quad (21)$$

one finds, e.g., for the spontaneous symmetry breaking (SSB) regime and $\overline{\nu} = 0$

$$\begin{aligned}
\frac{\partial \kappa}{\partial t} &= -(2 + \eta_\varphi) \kappa + \frac{1}{16\pi^2} \left\{ N^2 l_1^4(0) + 3l_1^4(2\lambda_1 \kappa) \right. \\
&\quad \left. + (N^2 - 1) \left[1 + \frac{\lambda_2}{\lambda_1} \right] l_1^4(\lambda_2 \kappa) - 4N_c \frac{h^2}{\lambda_1} l_1^{(F)4} \left(\frac{1}{N} h^2 \kappa \right) \right\} \\
\frac{\partial \lambda_1}{\partial t} &= 2\eta_\varphi \lambda_1 + \frac{1}{16\pi^2} \left\{ N^2 \lambda_1^2 l_2^4(0) + 9\lambda_1^2 l_2^4(2\lambda_1 \kappa) \right. \\
&\quad \left. + (N^2 - 1) [\lambda_1 + \lambda_2]^2 l_2^4(\lambda_2 \kappa) - 4 \frac{N_c}{N} h^4 l_2^{(F)4} \left(\frac{1}{N} h^2 \kappa \right) \right\} \\
\frac{\partial \lambda_2}{\partial t} &= 2\eta_\varphi \lambda_2 + \frac{1}{16\pi^2} \left\{ \frac{N^2}{4} \lambda_2^2 l_2^4(0) + \frac{9}{4} (N^2 - 4) \lambda_2^2 l_2^4(\lambda_2 \kappa) \right. \\
&\quad - \frac{1}{2} N^2 \lambda_2^2 l_{1,1}^4(0, \lambda_2 \kappa) + 3[\lambda_2 + 4\lambda_1] \lambda_2 l_{1,1}^4(2\lambda_1 \kappa, \lambda_2 \kappa) \\
&\quad \left. - 8 \frac{N_c}{N} h^4 l_2^{(F)4} \left(\frac{1}{N} h^2 \kappa \right) \right\}.
\end{aligned} \tag{22}$$

Here $\eta_\varphi = -\partial_t \ln Z_\varphi$, $\eta_\psi = -\partial_t \ln Z_q$ are the meson and quark anomalous dimensions, respectively. The symbols l_n^4 , l_{n_1, n_2}^4 and $l_n^{(F)4}$ denote mass threshold functions. A typical example is

$$l_n^4(w) = 8n\pi^2 k^{2n-4} \int \frac{d^4 q}{(2\pi)^4} \frac{\partial_t (Z_\varphi^{-1} R_k(q^2))}{[P(q^2) + k^2 w]^{n+1}} \tag{23}$$

with $P(q^2) = q^2 + Z_\varphi^{-1} R_k(q^2)$. These functions decrease monotonically with their arguments w and decay $\sim w^{-(n+1)}$ for $w \gg 1$. Since the arguments w are generally the (dimensionless) squared masses of the model, the main effect of the threshold functions is to cut off quantum fluctuations of particles with masses $M^2 \gg k^2$. Once the scale k is changed below a certain mass threshold, the corresponding particle no longer contributes to the evolution of the couplings and decouples smoothly. These threshold functions are non-perturbative in nature and are crucial for obtaining physically reasonable results as the system evolves into the far IR. Without them all (massive) mesons as well as the constituent quarks with masses $m_q = h(k) \sigma_R(k)$ in the SSB regime would continue to drive the evolution of couplings even for scales much smaller than their masses. A non-vanishing finite solution for the coupling constants and masses would then be impossible. One should, however, notice that there are threshold functions with vanishing arguments in (22). The reason is the existence of massless Goldstone bosons: the three pions for $N = 2$ and in addition the four kaons for $N = 3$. This is, of course, a consequence of neglecting the current quark masses. For $N = 2$ this problem can be circumvented by stopping the evolution of all couplings at $k = m_\pi$ by hand, thus mimicking the effect of an explicit χ SB.

The mass threshold functions are the main non-perturbative effect taken into account by approximating the solution of (16) with the truncation Γ_{eff} . It should be stressed that by “non-perturbative” we do not mean effects non-analytical in the coupling constants.

In fact, the dependence of the beta functions in (22) is analytical in all couplings of the linear σ -model. Effects of order $\exp(-1/\lambda^2)$ can not be captured explicitly within this model by truncations similar to the one proposed here. This does, however, not mean that such effects are excluded from our treatment of χ SB. Contributions to physical quantities of order $\exp(-1/\alpha_s)$ are certainly important for an understanding of low energy QCD and are most likely taken into account *implicitly* by our ansatz (5).

Finally a comment regarding the choice of the IR cutoff function R_k is in order. It is clear that (19) contains a certain degree of arbitrariness, since there is an infinite class of cutoff functions fulfilling (18). In addition, it is obvious that the numerical integration of the flow equations (22) and therefore also the results for physical observables will depend on the precise form of R_k . This situation is equivalent to the renormalization scheme dependence of ordinary perturbation theory. A full solution of (16) will be scheme or rather R_k independent in the limit $k \rightarrow 0$, though the trajectories along which the IR limit is reached might in principle depend on R_k . Once approximations are made to solve (16) this is no longer true and results often depend on the choice of R_k in the limit $k \rightarrow 0$. One may use this scheme dependence as a tool to obtain information about the robustness of a truncation of Γ_k by modestly varying R_k .

4 The chiral anomaly and the $O(4)$ -model

In (22) we have given the flow equations for the SSB regime in the limit $\overline{\nu} = 0$. Yet, the question remains to what extent one can expect this approximation to be a realistic one. Neglecting the effects of the chiral anomaly will result in an additional Goldstone boson, the η' which will artificially drive the running of the other couplings down to scales $k \simeq m_\pi$. On the other hand, from $m_{\eta'}^2 = \frac{N}{2}\nu\sigma_0^{N-2} \simeq 1 \text{ GeV}$ we infer $\nu \simeq 1 \text{ GeV}^2$ for $N = 2$. Thus, $\nu \rightarrow \infty$ appears to be a more realistic limit. In addition, one may ask if $N = 2$ or $N = 3$ is preferable. Certainly, in the real world, there are three light quark flavors and one might therefore be tempted to assume that $N = 3$ is the better choice. However, in the chiral limit the four K -mesons which are present for $N = 3$ are massless and will therefore artificially drive the evolution in the SSB regime where they should quickly decouple due to their comparably large masses. We conclude that in the chiral limit the two flavor case seems to be more appropriate to obtain a realistic picture of the IR world.

Fortunately, the two cases $N = 2$ and $\nu \rightarrow \infty$ go very well together. The deeper reason for this property is that for $N = 2$ the chiral group $SU_L(2) \times SU_R(2)$ is (locally) isomorphic to $O(4)$. Thus, the $(\mathbf{2}, \mathbf{2})$ representation φ of $SU_L(2) \times SU_R(2)$ may be decomposed into two real vector representations, (σ, π^k) and (η', a^k) of $O(4)$:

$$\varphi = \frac{1}{2}(\sigma - i\eta') + \frac{1}{2}(a^k + i\pi^k)\tau_k. \quad (24)$$

For $\nu \rightarrow \infty$ the masses of the η' and the a^k are easily seen to diverge and these particles decouple. We are then left with the original $O(4)$ symmetric linear σ -model of Gell-Mann and Levy [14] coupled to quarks. The flow equations of this model have been derived

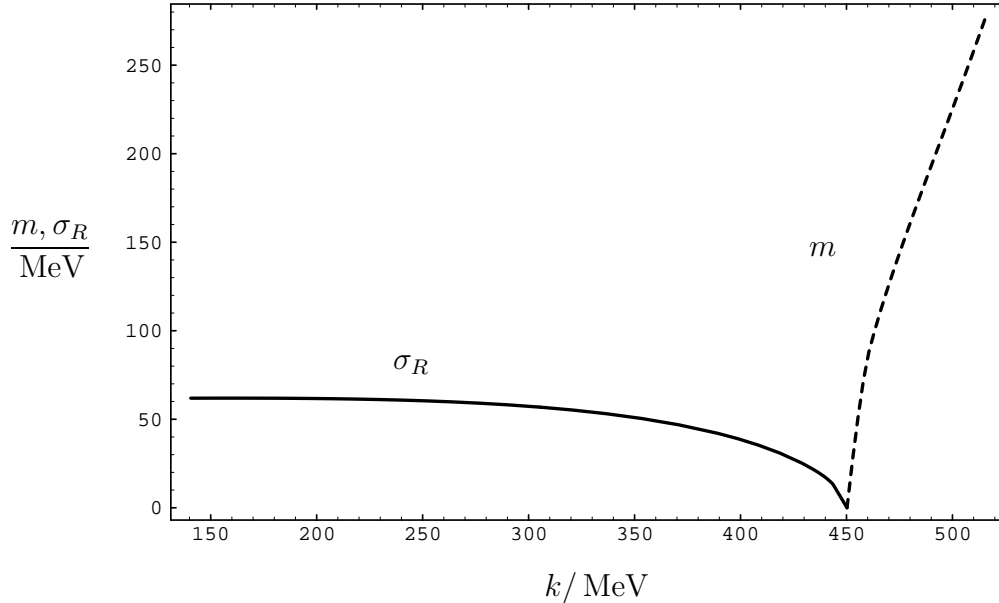


Figure 1: Evolution of the renormalized mass m in the symmetric regime (dashed line) and the vacuum expectation value $\sigma_R = Z_\varphi^{\frac{1}{2}} \sigma_0$ of the scalar field in the SSB regime (solid line) as functions of k for the $U_L(2) \times U_R(2)$ model. Initial values are $\lambda_1(k_\varphi) = \lambda_2(k_\varphi) = 0$ for $k_\varphi = 630$ MeV with $h^2(k_\varphi) = 300$ and $\tilde{\epsilon}_0 = 0.01$.

previously [8, 10] for the truncation of the effective action used here. Hence, we may compare the results for two different approximate implementations of the effects of the chiral anomaly:

- the $O(4)$ model corresponding to $N = 2$ and $\nu \rightarrow \infty$
- the $U_L(2) \times U_R(2)$ model corresponding to $N = 2$ and $\nu = 0$.

For reasons already mentioned we expect the first case to be closer to reality.

5 Results

Eqs. (22) and the corresponding set of flow equations for the symmetric regime constitute a coupled system of ordinary differential equations which can be integrated numerically. The most important result is that χ SB indeed occurs for a wide range of initial values of the parameters including the presumably realistic case of large renormalized Yukawa coupling and a bare mass $\overline{m}(k_\varphi)$ of order 100 MeV. A typical evolution of the renormalized mass $m(k)$ is plotted in figure 1. Driven by the strong Yukawa coupling, m decreases rapidly and goes through zero at a scale not far below k_φ . Here the system enters the SSB regime and a non-vanishing (renormalized) VEV σ_R for the meson field φ develops which turns out to

be reasonably stable already at scales $k \simeq m_\pi$ where the evolution has to be stopped by hand due to the vanishing pion mass in the chiral limit. We take this result as an indication that our truncation of the effective action Γ_k leads at least qualitatively to a satisfactory description of χ SB. The reason for the relative stability of the IR behavior of the VEV (and all other couplings) is that the quarks acquire a constituent mass $m_q = h\sigma_R \simeq 350$ MeV in the SSB regime. As a consequence they decouple once k becomes smaller than m_q and the evolution is then exclusively driven by the massless Goldstone bosons. This is also important in view of potential confinement effects expected to become important around $\Lambda_{\text{QCD}} \simeq 200$ MeV. Since confinement is not included in our model, one might be worried that such effects could spoil our results completely. Yet, the only particles here which should feel confinement are the colored quarks which are no longer important for the evolution of the system at scales around 200 MeV. One might therefore hope that an appropriate treatment of confinement is not crucial for this approach to χ SB.

More importantly, one finds that the system of flow equations exhibits an IR fixed point in the symmetric phase. As already pointed out one expects Z_φ to be rather small at the compositeness scale k_φ . In turn, one may assume that, at least for the initial range of running in the symmetric regime the mass parameter $\epsilon \sim Z_\varphi^{-1}$ is large. This means, in particular, that all threshold functions with arguments $\sim \epsilon$ may be neglected in this regime. As a consequence, the flow equations simplify considerably. We find, for instance, for the $U_L(2) \times U_R(2)$ model

$$\begin{aligned} \partial_t \tilde{\epsilon} &\equiv \partial_t \frac{\epsilon}{h^2} = -2\tilde{\epsilon} + \frac{N_c}{4\pi^2} \\ \partial_t \tilde{\lambda}_1 &\equiv \partial_t \frac{\lambda_1}{h^2} = \frac{N_c}{4\pi^2} h^2 \left[\frac{1}{2} \tilde{\lambda}_1 - \frac{1}{N} \right] \\ \partial_t \tilde{\lambda}_2 &\equiv \partial_t \frac{\lambda_2}{h^2} = \frac{N_c}{4\pi^2} h^2 \left[\frac{1}{2} \tilde{\lambda}_1 - \frac{2}{N} \right] \\ \partial_t h^2 &= \frac{N_c}{8\pi^2} h^4. \end{aligned} \tag{25}$$

This system possesses an attractive IR fixed point

$$\tilde{\lambda}_{1*} = \frac{1}{2} \tilde{\lambda}_{2*} = \frac{2}{N}. \tag{26}$$

Furthermore it is exactly soluble. The solution may be found in [9]. It can be seen that generally $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ approach their fixed point values long before the system enters the broken phase ($\epsilon \rightarrow 0$) and the approximation of large ϵ breaks down. Furthermore, h^2 only depends on the initial value

$$\tilde{\epsilon}_0 \equiv \frac{\epsilon(k_\varphi)}{h^2(k_\varphi)} = \frac{\overline{m}^2(k_\varphi)}{k_\varphi^2} Z_q^2(k_\varphi). \tag{27}$$

Hence, the system is approximately independent in the IR upon the initial values of λ_1 , λ_2 and h^2 , the only “relevant” parameter being $\tilde{\epsilon}_0$ once k_φ is specified. In other words,

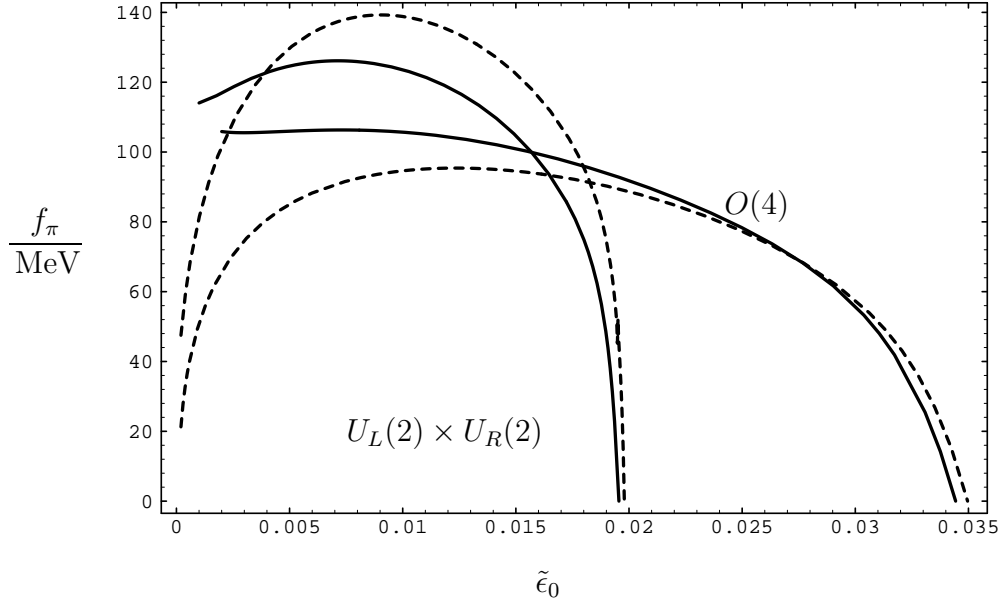


Figure 2: The pion decay constant f_π as a function of $\tilde{\epsilon}_0$ for $k_\varphi = 630$ MeV, $\lambda_1(k_\varphi) = \lambda_2(k_\varphi) = 0$ and $h^2(k_\varphi) = 300$ (solid line) as well as $h^2(k_\varphi) = 10^4$ (dashed line).

the effective action loses almost all its “memory” in the far IR of where in the UV it came from. This feature of the flow equations leads to a perhaps surprising or unexpected degree of predictive power which will be especially useful once the current quark masses are included. In addition, also the dependence of IR quantities like f_π on $\tilde{\epsilon}_0$ is not very strong as shown in figure 2. The two remaining parameters $\tilde{\epsilon}_0$ and k_φ can be fixed by using $f_\pi \equiv 2\sigma_R(k=0) \simeq 93$ MeV and $m_q \equiv (h\sigma_R)(k=0) \simeq 350$ MeV as phenomenological input. One obtains for the $O(4)$ model

$$\begin{aligned} \tilde{\epsilon}_0 &\simeq 0.02 \\ k_\varphi &\simeq 650 \text{ MeV} \simeq \left(\frac{1}{3} \text{ fm}\right)^{-1}. \end{aligned} \quad (28)$$

We note that the result for k_φ is quite encouraging. Let us recall that k_φ is the compositeness scale, i.e. the scale at which the QCD vacuum structure which is supposed to be responsible for the formation of mesonic bound states should become “visible” for the block-spin RG (16). One would therefore expect that the length scale $1/k_\varphi$ should at least roughly agree with corresponding length scales of successful models of the QCD vacuum. This is indeed the case: The average instanton size in the instanton liquid model [15, 16] and the vacuum correlation length of the stochastic vacuum model of QCD [17] both are in good agreement with our result of $\frac{1}{3}$ fm. We have furthermore used the results (28) for an estimate of the chiral condensate:

$$|\langle \bar{q}q \rangle|^{\frac{1}{3}} \simeq 200 \text{ MeV} \quad (29)$$

which is in good agreement with results, e.g., from chiral perturbation theory [2]. This result is non-trivial, since $\langle \bar{q}q \rangle = -\tilde{\epsilon}_0 Z_\varphi^{-\frac{1}{2}}(k=0) f_\pi k_\varphi^2$. Hence, not only k_φ and f_π enter but also the IR value of Z_φ .

6 Conclusions

We have used a QCD-motivated extended Nambu–Jona-Lasinio model in its bosonized form (a linear σ -model of QCD coupled to quarks) to study χ SB. The main technical tool for this intrinsically non-perturbative problem was the exact renormalization group equation (16) for the “block-spin” effective action Γ_k in the continuum. Already a crude truncation of Γ_k , keeping only naively relevant and marginal operators, leads to qualitatively and also quantitatively satisfactory results. The numerical integration of (16) revealed the following picture of χ SB:

- Light mesons form, presumably due to non-perturbative QCD interactions, around a scale $k_\varphi \simeq 650$ MeV which is in good agreement with typical scales of QCD vacuum models.
- At the scale k_φ and somewhat below the system is still in the chirally symmetric regime even though there are already mesonic bound states. χ SB takes place at scales $k \simeq (400 - 500)$ MeV due to a strong initial Yukawa coupling between quarks and mesons which drives the mass parameter negative.
- For large initial Yukawa coupling the evolution of the model in the symmetric regime is governed by a fixed point. The IR results are therefore almost insensitive to most initial conditions on the coupling constants thus enhancing greatly the predictive power of the model.
- Reasonable values for the constituent quark mass around $m_q \simeq 350$ MeV, $f_\pi \simeq 100$ MeV and the chiral condensate $|\langle \bar{q}q \rangle|^{1/3} \simeq 200$ MeV can be obtained.

We consider these results as encouraging support for the viability of the model itself as well as the truncations described in this work. There are several directions of straightforward improvement or generalization:

- The effects of the chiral anomaly should be taken into account more accurately by allowing for a non-vanishing but finite ν .
- Current quark masses may be included to linear order by extending the truncation of the effective action with a term $\sim \text{tr } \varphi^\dagger \mathcal{M} + \text{tr } \mathcal{M}^\dagger \varphi$ with $\mathcal{M} = \text{diag}(m_u, m_d, \dots)$. As explained earlier this should be accompanied by adding the strange quark as a third light flavor. Due to the IR fixed point behavior in the symmetric regime this will allow to “predict” all pseudoscalar and scalar masses and mixing angles as well as the corresponding decay constants with only a few input parameters.

- Additional terms should be included in the effective potential. Already for $\overline{\nu} = 0$ there are indications for a first order phase transition in the mass parameter for $N = 3$. The numerical analysis shows that λ_1 can turn negative, signaling the importance of higher dimensional operators to stabilize the potential. The inclusion of several such operators is also required by a consistent computation of all pseudoscalar meson masses and mixing angles to linear order in the current quark masses.
- An extension to finite temperature is straightforward [18]. It simply amounts to the replacement $\int \frac{d^d q}{(2\pi)^d} \rightarrow T \sum_n \int \frac{d^{d-1}}{(2\pi)^{d-1}}$ in all threshold functions. This should allow for a determination of the critical temperature T_c and might help to shed light on the nature of the chiral phase transition [19, 20]. In addition one could hope to answer the question if there are mesonic bound states above T_c .

Last but not least, one might try to attempt to “derive” the initial conditions for the flow of the linear σ -model directly from QCD following the lines of [12, 21]. Even though this appears to be principally feasible it will still require a significant amount of preparatory work. The results presented in this talk should encourage to follow this road.

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